MATH6031 Lecture 3

Last time: We rewrote the extended Duflo isom. as

$$I_{PBW} \circ J^{\frac{1}{2}} : H^\bullet(\mathfrak{g}, S^\bullet(\mathfrak{g})) \xrightarrow{\sim} HH^\bullet(U^\bullet(\mathfrak{g}), U^\bullet(\mathfrak{g}))$$

§ Complex manifolds

- almost complex manifold: a smooth manifold M equipped with $J : TM \to TM$ (here TM is the tangent bundle of the smooth manifold M) s.t. $J^2 = -Id$.

$$\rightsquigarrow \quad T_{\mathbb{C}}M = TM \otimes_{\mathbb{R}} \mathbb{C} = T' \oplus T'' \quad \left(\text{more commonly denoted as } T^{10} \oplus T^{01}\right)$$

where T' (resp. T'') is the eigenbundle corr. to the eigenvalue i (resp. $-i$).

- M is a complex manifold iff J is integrable

$$\overset{\Delta}{\iff} [T', T'] \subset T' \quad \left(\text{or } [T'', T''] \subset T''\right)$$

$$\iff \bar{\partial}^2 = 0 \quad (\text{or } \partial^2 = 0)$$

where $\bar{\partial} : \Omega^{\bullet,\bullet}(M) \to \Omega^{\bullet,\bullet+1}(M)$

$\partial : \Omega^{\bullet,\bullet}(M) \to \Omega^{\bullet+1,\bullet}(M)$

are the usual Dolbeault operators on the bigraded complex $\Omega^{\bullet,\bullet}(M)$ given by

$$\Omega^{p,q}(M) = \Gamma(M, \overbrace{\Lambda^p (T')^* \otimes \Lambda^q (T'')^*}^{\Omega^{p,q}_M})$$

We have $d = \partial + \bar{\partial}$.

If $\bar{\partial}^2 = 0$, then

$$(\Omega^{\bullet,\bullet}(M), \bar{\partial}, \wedge)$$

is a differential graded algebra (DGA) called the

Dolbeault algebra of M; its cohomology $H^{\cdot}_{\bar\partial}(M)$ is called the Dolbeault cohomology of M.

- Given a smooth complex vector bundle E, we similarly have

$$\Omega^{p,q}(M,E) = \Gamma(M, \Lambda^p(T')^* \otimes \Lambda^q(T'')^* \otimes E)$$

E is a holomorphic vector bundle

$\Longleftrightarrow$ $\exists$ a $\bar\partial$-connection

$$\bar\partial : \Gamma(M,E) \longrightarrow \Omega^{0,1}(M,E)$$

s.t. $\underset{\text{on } \Gamma(M,E)}{\bar\partial}(fs) = (\underset{\Gamma(M)}{\bar\partial} f) \otimes s + f \cdot \underset{\text{on } \Gamma(M,E)}{\bar\partial} s$ for $f \in C^\infty(M)$ and $s \in \Gamma(M,E)$

and $\bar\partial^2 = 0$.

In this case, we have

$$(\Omega^{0,\cdot}(M,E), \bar\partial)$$

which is differential graded (DG) module over the DGA $(\Omega^{0,\cdot}(M), \bar\partial, \wedge)$.

We also have the Dolbeault cohomology $H^{\cdot}_{\bar\partial}(M,E)$.

§ Deformation theoretical meaning of $H^{\cdot}_{\bar\partial}(M,E)$

- $H^0_{\bar\partial}(M,E) = \ker \bar\partial =$ space of global holomorphic sections of E.

- For a smooth complex vector bundle E, two $\bar\partial$-connections $\bar\partial_1, \bar\partial_2$ differ by a $(0,1)$-form with values in End E, i.e.

$$\boxed{\bar\partial_2 - \bar\partial_1 = \xi} \in \Omega^{0,1}(M, \mathrm{End}\, E)$$

$\Rightarrow$ {space of [illegible]} is an affine space on $\Omega^{0,1}(M, \mathrm{End}\, E)$

$\Rightarrow \left\{\begin{matrix}\text{space of} \\ \bar\partial\text{-connections} \\ \text{on } E\end{matrix}\right\}$ is an affine space on $\Omega^{0,1}(M, \operatorname{End} E)$

Now $\bar\partial_2 = \bar\partial_1 + \xi$ and $\bar\partial_1^2 = \bar\partial_2^2 = 0$

$$\Rightarrow \quad 0 = \bar\partial_2^2 = (\bar\partial_1 + \xi)^2 = \cancel{\bar\partial_1^2} + \bar\partial_1 \circ \xi + \xi \circ \bar\partial_1 + \xi^2$$

So any infinitestimal deformation $\bar\partial_\varepsilon$ of $\bar\partial$ (meaning that $\bar\partial_\varepsilon \equiv \bar\partial \pmod{\varepsilon}$)

can be written as

$$\boxed{\bar\partial_\varepsilon = \bar\partial + \varepsilon \cdot \xi} \quad \text{where } \xi \in \Omega^{0,1}(M, \operatorname{End} E)$$

s.t. $\bar\partial \circ \xi + \xi \circ \bar\partial = 0$

$$\Longleftrightarrow \quad \bar\partial_{\operatorname{End} E}(\xi) = 0 \quad \Rightarrow [\xi] \in H^1_{\bar\partial}(M, \operatorname{End} E)$$

<u>Rmk</u> If E is a holomorphic vector bundle with a $\bar\partial$-connection $\bar\partial_E$, then

$$\bar\partial_{\operatorname{End} E}(s) := \bar\partial_E \circ s - s \circ \bar\partial_E \quad \text{for } s \in \Gamma(M, \operatorname{End} E)$$

defines a $\bar\partial$-connection on $\operatorname{End}(E)$.

Furthermore, an infinitestimal deformation $\bar\partial_\varepsilon = \bar\partial + \varepsilon \cdot \xi$ is trivial

$\Longleftrightarrow$ $\bar\partial_\varepsilon$ can be identified with $\bar\partial$ by an automorphism of E of the form $\operatorname{Id} + \varepsilon \cdot s$ for some $s \in \Gamma(\operatorname{End} E)$

$\Longleftrightarrow$ $\exists\, s \in \Gamma(\operatorname{End} E)$ s.t. $\boxed{\bar\partial \circ s - s \circ \bar\partial = \xi}$

$\Longleftrightarrow$ $\bar\partial_{\operatorname{End} E}(s) = \xi$

Conclusion: $H^1_{\bar\partial}(M, \operatorname{End} E) = \left\{\begin{matrix}\text{equiv. classes of} \\ \text{infinitestimal deform}^{\text{ns}} \\ \text{of the holom. str. on } E\end{matrix}\right\}$

of the holom. str. on E]

- obstructions for extending $\bar\partial_\varepsilon = \boxed{\bar\partial + \varepsilon\cdot\xi}$ to an all-order deformation of the holom. str. on E lie in $H^2_{\bar\partial}(M, \mathrm{End}\, E)$: in general, $\bar\partial + \xi$ defines a (new) holom. str. on E $\Leftrightarrow (\bar\partial + \xi)^2 = 0$

$$\bar\partial + \varepsilon\xi_1 + \varepsilon^2\xi_2 \qquad\qquad \cancel{\bar\partial^2}^{\,0} + \underbrace{\bar\partial\cdot\xi + \xi\cdot\bar\partial}_{\bar\partial_{\mathrm{End}\,E}\xi} + \underbrace{\xi^2}_{\frac12[\xi,\xi]}$$

$\bar\partial$-closed

$$\bar\partial\,\underline{\underline{\xi_2}} + \underbrace{\tfrac12[\xi_1,\xi_1]}_{H^2_{\bar\partial}(M,\mathrm{End}\,E)} = 0 \qquad \Leftrightarrow \qquad \bar\partial_{\mathrm{End}\,E}\,\xi + \frac12[\xi,\xi] = 0$$

The last equation is called the Maurer-Cartan equation associated to the deformations of holom. str. on E.

§ Atiyah and Todd classes

Consider a holom. vector bundle $E \to M$.

Choose a connection

$$\nabla : \Gamma(M, E) \longrightarrow \Omega^1(M, E)$$

compatible with the holomorphic structure on E

this means that the (0,1)-part of ∇ i.e. ∇'' (when we write $\nabla = \underset{(1,0)}{\nabla'} + \underset{(0,1)}{\nabla''}$) is the $\bar\partial$-conn. on E.

Then the curvature of ∇ is of the form

$$R := \nabla^2 = (\nabla' + \nabla'')^2 \overset{\because (\nabla'')^2 = 0}{=\!=} \underline{R^{20}} + R^{11}$$

Note that $\nabla'' = \bar\partial \Leftrightarrow$ we can write

$$\underline{\nabla = d + \Gamma} \quad \text{(locally)}$$

where $\Gamma \in \Omega^{1,0}(\mathrm{End}\, E)$

Therefore, $\quad R'' = \bar\partial \Gamma$ (locally)

$$\Rightarrow \quad \bar\partial R'' = 0$$

$$\Rightarrow \quad [R''] \in H^1_{\bar\partial}((T')^* \otimes \operatorname{End} E)$$

Def The Atiyah class of E is defined as

$$\mathrm{at}_E := [R''] \in H^1_{\bar\partial}((T')^* \otimes \operatorname{End} E) = H^1_{\bar\partial}(\operatorname{Hom}(T', \operatorname{End} E))$$

As in Chern-Weil theory, we can show that at_E is independent of the choice of the connection ∇.

For any $n \in \mathbb{Z}_{\geq 0}$, define the n-th scalar Atiyah class $a_n(E)$ as

$$a_n(E) := \operatorname{tr}(\mathrm{at}_E^n) \in H^n_{\bar\partial}(M, \underbrace{\Lambda^n (T')^*}_{\text{holom. poly-vector fields}})$$

Def The Todd class of E is defined as

$$\mathrm{td}_E := \det\left(\frac{\mathrm{at}_E}{1 - e^{-\mathrm{at}_E}}\right)$$

expressed as a formal series in $a_n(E)$.

Rmk : If M is Kähler, then $\frac{i}{2\pi} a_1(E) = c_1(E)$.

§ Hochschild cohomology of a smooth mfd

M : smooth mfd.

- $T^\bullet_{poly} M := \Gamma(M, \Lambda^\bullet TM)$

  equipped with product $\wedge$ and differential $d = 0$

  So $(T^\bullet_{poly} M, 0, \wedge)$ is a DGA with trivial differential

DG subalgebra

- $D_{poly}M \subset (C(C^\infty(M), C^\infty(M)), d_H, \cup)$

Hochschild complex

consisting of Hochschild cochains which are differential operators in each argument.

e.g. $X = \mathbb{C}^n = \operatorname{Spec} \mathbb{C}[x_1, \dots, x_n]$

$X = \operatorname{Spec} A$

$T^{\bullet}_{poly} X \cong HH^{\bullet}(A)$

Thm (Vey 1975; C^∞-version of HKR) 1962

The degree 0 graded map

$$I_{HKR} : (T^{\bullet}_{poly}M, \wedge) \longrightarrow (D^{\bullet}_{poly}M, d_H)$$

$$v_1 \wedge \dots \wedge v_n \longmapsto \Big(f_1 \otimes \dots \otimes f_n \longmapsto \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} (-1)^{|\sigma|} v_{\sigma(1)}(f_1) \dots v_{\sigma(n)}(f_n) \Big)$$

is a quasi-isom of complexes which induces an isom of graded algebras in cohomology.

Pf:
- check that I_{HKR} is a morphism of complexes.
- check that $\wedge, \cup, d_H, I_{HKR}$ are all $C^\infty(M)$-linear
- Observation: $D^{\bullet}_{poly}M$ = Hochschild complex of the algebra J^∞_M : ∞-jets of functions of M

$J^\infty_M = \operatorname{Hom}_{C^\infty(M)}(D^1_{poly}M, C^\infty(M))$

with product $(j_1 \cdot j_2)(P) = (j_1 \otimes j_2)(\Delta(P))$

where $\Delta(P) \in D^2_{poly}M$ is defined by

$\Delta(P)(f, g) = P(fg)$.

As an algebra, we have

$$J^\infty_M \cong \Gamma(M, \hat{S}(T^*M))$$

Then the statement follows from a previous result (Lemma 2.6). #

If V is a finite-dim$^{\text{nl}}$ vect. space, then

$\Lambda^{\bullet} V^* \hookrightarrow C^{\bullet}(\hat{S}(V), k)$ is a quasi-isom which

induces an isom of graded algebras
$$\Lambda^\bullet V^* \cong HH^\bullet(\hat{S}(V), k)$$

§ Hochschild cohomology of a complex mfd

M : cpx mfd

E : any vector bdle

- $T'^{\bullet}_{poly}(M,E) := \Gamma(M, E \otimes \Lambda^\bullet T')$

- Define ∂-differential operators as elts of $\mathrm{End}(C^\infty(M))$ generated by functions and type $(1,0)$ vector fields. If we have a bundle E, we similarly define E-valued ∂-differential operators as linear maps $C^\infty(M) \to \Gamma(M,E)$ which are compositions of ∂-diff. operators with either sections of E or $\underbrace{T' \otimes E}_{\varphi}$

as E-valued type $(1,0)$ vector fields

$$D'^{\bullet}_{poly}(M,E) \xhookrightarrow{\text{DG subcomplex}} (C^\bullet(C^\infty(M), \Gamma(M,E)), d_H)$$

consisting of cochains that are ∂-differential operators in each argument.

Thm (HKR) The degree 0 graded map
$$I_{HKR} : (T'^{\bullet}_{poly}(M,E), 0) \longrightarrow (D'^{\bullet}_{poly}(M,E), d_H)$$
$$(v_1 \wedge \cdots \wedge v_n) \otimes s \mapsto \Big(f_1 \otimes \cdots \otimes f_n \mapsto \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} (-1)^{|\sigma|} v_{\sigma(1)}(f_1) \cdots v_{\sigma(n)}(f_n)\, s\Big)$$
is a quasi-isom. of complexes.

Finally, let us consider the special case

$$E = \Lambda^\bullet (T'')^*$$

Then $T'^{\bullet}_{poly}(M, \Lambda^\bullet (T'')^*) = \Omega^{0,\bullet}(M, \Lambda^\bullet T')$

is a DGA with product $\wedge$ and differential $\bar\partial$ which satisfy $\bar\partial(v \wedge w) = \bar\partial(v) \wedge w + (-1)^{|v|} v \wedge \bar\partial(w)$

On the other hand, $\bar\partial$ acts on a ∂-diff. operator P by

$$(\bar\partial(P))(f) = \bar\partial(P(f)) - P(\bar\partial f)$$

This extends uniquely to a degree 1 derivation on $D'^{\bullet}_{poly}(M, \Lambda^\bullet (T'')^*)$, making it a DGA with product

$$(P \cup Q)(f_1, \dots, f_{m+n}) = (-1)^{m|Q|} P(f_1, \dots, f_m) Q(f_{m+1}, \dots, f_{m+n})$$

where $|\cdot|$ refers to the exterior degree.

$\rightsquigarrow$ $I_{HKR} : (\Omega^{0,\bullet}(M, \Lambda^\bullet T'), \bar\partial) \xrightarrow{\sim} (D'^{\bullet}_{poly}(M, \Lambda^\bullet (T'')^*), d_H + \bar\partial)$

is a quasi-isom.

$\rightsquigarrow$ $I_{HKR} : H_{\bar\partial}(\Lambda^\bullet T') \cong H^\bullet(D'^{\bullet}_{poly}(M, \Lambda^\bullet (T'')^*), d_H + \bar\partial)$

is an isom of (graded) vector spaces.
BUT the product is not preserved.

<u>Thm</u> (Kontsevich) The map $I_{HKR} \circ td^{\frac{1}{2}}_{T'}$ induces an isom of graded algebras

$$H_{\bar\partial}(\Lambda^\bullet T') \xrightarrow{\cong} H^\bullet(D'^{\bullet}_{poly}(M, \Lambda^\bullet (T'')^*), d_H + \bar\partial)$$

on cohomology.

deg 0 : $T'^{\bullet}_{poly}(M) \xrightarrow{\cong} H^\bullet(D'^{\bullet}_{poly}(M), d_H)$